Chapter review

1 a $v = \frac{dx}{dt} = 4e^{0.5t}$

$x = \int 4e^{0.5t}\,dt = 8e^{0.5t} + A$

When $t = 0$, $x = 0$

$0 = 8 + A \Rightarrow A = -8$

$x = 8e^{0.5t} - 8 = 8(e^{0.5t} - 1)$

b $a = \frac{dv}{dt} = 2e^{0.5t}$

When $t = \ln 9$

$a = 2e^{0.5\ln 9} = 2e^{\ln 3} = 2 \times 3 = 6$

The acceleration of P when $t = \ln 9$ is $6\,\text{m s}^{-2}$

2 a $a = \frac{dv}{dt} = 20t\,e^{-t^2}$

$v = \int 20t\,e^{-t^2}\,dt = -10e^{-t^2} + A$

When $t = 0$, $v = 8$

$8 = -10 + A \Rightarrow A = 18$

$v = 18 - 10e^{-t^2}$

b As $t \to \infty$, $e^{-t^2} \to 0$ and $v \to 18$

The limiting velocity of P is $18\,\text{m s}^{-1}$

3 $a = \frac{dv}{dt} = \frac{18}{(2t+3)^3} = 18(2t+3)^{-3}$

$v = \int 18(2t+3)^{-3}\,dt = \frac{18}{-2 \times 2}(2t+3)^{-2} + A = A - \frac{9}{2(2t+3)^2}$

When $t = 0$, $v = 0$

$0 = A - \frac{9}{2 \times 3^2} \Rightarrow A = \frac{1}{2}$

$v = \frac{1}{2} - \frac{9}{2(2t+3)^2}$

When $v = 0.48$

$0.48 = \frac{1}{2} - \frac{9}{2(2t+3)^2} \Rightarrow \frac{9}{2(2t+3)^2} = 0.02$

So $(2t+3)^2 = \frac{9}{2 \times 0.02} = 225 \Rightarrow 2t + 3 = \pm 15 \Rightarrow t = 6$ or $t = -9$

As $t \geqslant 0$, the solution is $t = 6$

4 a $a = \frac{dv}{dt} = \frac{100}{(2t+5)^2} = 100(2t+5)^{-2}$

$v = \int 100(2t+5)^{-2}\,dt = \frac{100}{2\times -1}(2t+5)^{-1} + A = A - \frac{50}{2t+5}$

When $t = 0$, $v = 0$

$0 = A - \frac{50}{5} \Rightarrow A = 10$

$v = 10 - \frac{50}{2t+5}$

b $v = \frac{dx}{dt} = 10 - \frac{50}{2t+5}$

$x = \int \left(10 - \frac{50}{2t+5}\right) dt = 10t - 25\ln(2t+5) + B$

When $t = 0$, $x = 0$

$0 = -25\ln 5 + B \Rightarrow B = 25\ln 5$

$x = 10t - 25\ln(2t+5) + 25\ln 5$

When $t = 10$

$x = 100 - 25\ln 25 + 25\ln 5 = 100 - 25\ln\frac{25}{5} = 100 - 25\ln 5 = 59.8\,\text{m (3 s.f.)}$

The distance moved by the car in the first 10 seconds of its motion is $(100 - 25\ln 5)$ m.

5 a $a = \frac{dv}{dt} = \cos^2 t = \frac{1}{2} + \frac{1}{2}\cos 2t$

$v = \int \left(\frac{1}{2} + \frac{1}{2}\cos 2t\right) dt = \frac{1}{2}t + \frac{1}{4}\sin 2t + A$

When $t = 0$, $v = 0$

$0 = 0 + 0 + A \Rightarrow A = 0$

$v = \frac{1}{2}t + \frac{1}{4}\sin 2t$

When $t = \pi$, $v = \frac{\pi}{2} + \frac{1}{4}\sin 2\pi = \frac{\pi}{2} + 0 = \frac{\pi}{2}\,\text{m s}^{-1}$

b $x = \int v\,dt = \int \frac{1}{2}t + \frac{1}{4}\sin 2t\,dt = \frac{1}{4}t^2 - \frac{1}{8}\cos 2t + B$

When $t = 0$, $x = 0$

$0 = 0 - \frac{1}{8} + B \Rightarrow B = \frac{1}{8}$

$v = \frac{1}{4}t^2 - \frac{1}{8}\cos 2t + \frac{1}{8}$

When $t = \frac{\pi}{4}$, $x = \frac{\pi^2}{64} - \frac{1}{8}\cos\frac{\pi}{2} + \frac{1}{8} = \frac{\pi^2}{64} + \frac{1}{8} = \frac{1}{64}(\pi^2 + 8)\,\text{m}$ as required.

6 a When $t = 2.5, v = \frac{1}{2}t^2$

$a = \frac{dv}{dt} = t$

So when $t = 2.5, a = 2.5$, i.e. the acceleration of P is $2.5\,\text{m s}^{-2}$ in the direction of x-increasing

b When $t = 5, v = 8e^{4-t}$

$a = \frac{dv}{dt} = -8e^{4-t}$

So when $t = 5, a = -8e^{-1}$, i.e. the acceleration of P is $8e^{-1}\,\text{m s}^{-2}$ in the direction of x-decreasing

c The distance of P from O when $t = 6$ is given by

$$x = \int_0^4 \frac{1}{2}t^2\,dt + \int_4^6 8e^{4-t}\,dt = \left[\frac{t^3}{6}\right]_0^4 + \left[-8e^{4-t}\right]_4^6$$

$$= \frac{64}{6} - 8e^{-2} + 8 = \frac{56}{3} - 8e^{-2}$$

The distance of P from O when t = e is $\left(\frac{56}{3} - 8e^{-2}\right)\text{m} = 17.6\,\text{m}$ (3 s.f.)

7 a $a = \frac{dv}{dt} = \frac{2t+3}{t+1} = 2 + \frac{1}{t+1}$

$v = \int a\,dt = 2t + \ln(t+1) + A$

When $t = 0, v = 0$

$0 = 0 + A \Rightarrow A = 0$

So $v = 2t + \ln(t+1)$

b $x = \int v\,dt = \int 2t + \ln(t+1)\,dt$

Integrate $\ln(t + 1)$ using integration by parts

$$\int \ln(t+1)\,dt = \int 1\ln(t+1)\,dt = t\ln(t+1) - \int \frac{t}{t+1}\,dt$$

$$= t\ln(t+1) - \int\left(1 - \frac{1}{t+1}\right)dt = t\ln(t+1) - t + \ln(t+1) + C$$

$$= (t+1)\ln(t+1) - t + C$$

So $x = t^2 + (t+1)\ln(t+1) - t + C$

When $t = 0, x = 0$

$0 = 0 + 0 - 0 + C \Rightarrow C = 0$

So $x = t^2 + (t+1)\ln(t+1) - t$

When $t = 2$

$x = 4 + 3\ln 3 - 2 = (2 + 3\ln 3)\,\text{m}$

8 **a** Solving $18-\frac{t^2}{5}=0$ gives

$18=\frac{t^2}{5}\Rightarrow t^2=90\Rightarrow t=\pm\sqrt{90}=\pm 3\sqrt{10}$

As $T>0, T=3\sqrt{10}$ s

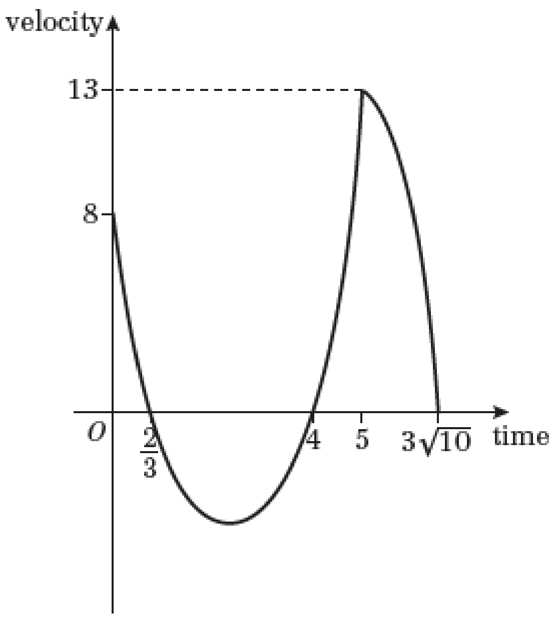
b For $0\leqslant t\leqslant 5$, the curve $y=3t^2-14t+8$ is a parabola with a positive t^2 coefficient

When $t = 5$, $y = 75 - 70 + 8 = 13$, so the curve connects (0, 8) and (5, 13)

$3t^2-14t+8=(3t-2)(t-4)$, so the curve cuts the x-axis at $\left(\frac{2}{3},0\right)$ and $(4,0)$

For $5<t\leqslant T$, the curve $y=18-\frac{t^2}{5}$ is a parabola with a negative t^2 coefficient

From part **a**, it connects (5, 13) and $(3\sqrt{10},0)$



c The graph has a minimum point where $\frac{d}{dt}(3t^2-14t+8)=0$

$6t-14=0\Rightarrow t=\frac{7}{3}$

The acceleration is positive for $\frac{dv}{dt}>0$, so the solution is $\frac{7}{3}<t<5$

8 d The total distance travelled is given by

$$\int_0^{\frac{2}{3}}\left(3t^2-14t+8\right)\,dt+\left|\int_{\frac{2}{3}}^{4}\left(3t^2-14t+8\right)\,dt\right|$$

$$+\int_4^5\left(3t^2-14t+8\right)\,dt+\int_5^{3\sqrt{10}}\left(18-\frac{t^2}{5}\right)dt$$

$$=\left[t^3-7t^2+8t\right]_0^{\frac{2}{3}}+\left|\left[t^3-7t^2+8t\right]_{\frac{2}{3}}^{4}\right|$$

$$+\left[t^3-7t^2+8t\right]_4^5+\left[18t-\frac{t^3}{15}\right]_5^{3\sqrt{10}}$$

$$=\left(\frac{68}{27}-0\right)+\left|\left(-16-\frac{68}{27}\right)\right|+\left(-10-(-16)\right)$$

$$+\left(18\sqrt{90}-\frac{90\sqrt{90}}{15}-\left(90-\frac{25}{3}\right)\right)$$

$$=\frac{68}{27}+\frac{500}{27}+\frac{162}{27}+18\sqrt{90}-6\sqrt{90}-\frac{245}{3}$$

$$=12\sqrt{90}-\frac{1475}{27}$$

$$=59.2$$

9 a $v=(t-4)(3t-8)$

$v=3t^2-20t+32$

$a=\frac{dv}{dt}=6t-20$

b Distance travelled

$$=\int_0^{\frac{8}{3}}\left(3t^2-20t+32\right)\,dt+\left|\int_{\frac{8}{3}}^{3}\left(3t^2-20t+32\right)\,dt\right|$$

$$=\left[t^3-10t^2+32t\right]_0^{\frac{8}{3}}+\left|\left[t^3-10t^2+32t\right]_{\frac{8}{3}}^{3}\right|$$

$$=\left(\frac{512}{27}-\frac{640}{9}+\frac{256}{3}\right)+\left|\left(33-\frac{896}{27}\right)\right|$$

$$=\frac{896}{27}+\frac{5}{27}$$

$$=\frac{901}{27}$$

9 c $x = \int v\,\mathrm{d}t = \int \left(3t^2 - 20t + 32\right)\,\mathrm{d}t$

$x = t^3 - 10t^2 + 32t + c$

At $t = 0,\ x = 0$, so $c = 0$

$x = t^3 - 10t^2 + 32t$

$x = t\left(t^2 - 10t + 32\right)$

If $t^2 - 10t + 32 = 0$ has a solution, then $'b^2 - 4ac' \geqslant 0$.

However,

$b^2 - 4ac = 100 - 4 \times 1 \times 32 = -28 < 0$

So $t^2 - 10t + 32 = 0$ has no solutions.

Therefore P never returns to O.

10 a $a = 3t - 4$

$v = \int (3t - 4)\,\mathrm{d}t = \frac{3t^2}{2} - 4t + c$

When $t = 0,\ v = 2,$ so $c = 2$ $\frac{1}{2}\ln\left(\frac{5}{8}\right) = -kX$

$v = \frac{3t^2}{2} - 4t + 2$

Solving $\frac{3t^2}{2} - 4t + 2 = 0$

$3t^2 - 8t + 4 = 0$

$(3t - 2)(t - 2) = 0$

$t = \frac{2}{3}$ or $t = 2$

b Distance travelled

$$= \int_0^{\frac{2}{3}} \left(\frac{3t^2}{2} - 4t + 2\right)\mathrm{d}t + \left|\int_{\frac{2}{3}}^{2} \left(\frac{3t^2}{2} - 4t + 2\right)\mathrm{d}t\right| + \int_2^4 \left(\frac{3t^2}{2} - 4t + 2\right)\mathrm{d}t$$

$$= \left[\frac{t^3}{2} - 2t^2 + 2t\right]_0^{\frac{2}{3}} + \left|\left[\frac{t^3}{2} - 2t^2 + 2t\right]_{\frac{2}{3}}^{2}\right| + \left[\frac{t^3}{2} - 2t^2 + 2t\right]_2^4$$

$$= \left[\frac{\left(\frac{2}{3}\right)^3}{2} - 2\left(\frac{2}{3}\right)^2 + 2\left(\frac{2}{3}\right)\right] - \left[\left(\frac{(2)^3}{2} - 2(2)^2 + 2(2)\right) - \left(\frac{\left(\frac{2}{3}\right)^3}{2} - 2\left(\frac{2}{3}\right)^2 + 2\left(\frac{2}{3}\right)\right)\right]$$

$$+ \left[\left(\frac{(4)^3}{2} - 2(4)^2 + 2(4)\right) - \left(\frac{(2)^3}{2} - 2(2)^2 + 2(2)\right)\right]$$

$$= \frac{248}{27}$$

11 $a = \dfrac{6}{x^2} = 6x^{-2}$

$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{2}v^2\right) = 6x^{-2}$

$\dfrac{1}{2}v^2 = \int 6x^{-2}\,\mathrm{d}x = \dfrac{6x^{-1}}{-1} + A$

$\dfrac{1}{2}v^2 = A - \dfrac{6}{x}$

At $x = 3,\ v = 4$

$\dfrac{1}{2} \times 16 = A - 2 \Rightarrow A = 10$

$\dfrac{1}{2}v^2 = 10 - \dfrac{6}{x}$

$v^2 = 20 - \dfrac{12}{x}$

$v = \sqrt{\left(20 - \dfrac{12}{x}\right)}$

12 $a = -\left(3 + \dfrac{1}{4}x\right)$

$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{2}v^2\right) = -3 - \dfrac{1}{4}x$

$\dfrac{1}{2}v^2 = \int\left(-3 - \dfrac{1}{4}x\right)\mathrm{d}x = -3x - \dfrac{1}{8}x^2 + A$

At $x = 0,\ v = 8$

$32 = -0 - 0 + A \Rightarrow A = 32$

$\dfrac{1}{2}v^2 = 32 - 3x - \dfrac{1}{8}x^2$

When $v = 0$

$0 = 32 - 3x - \dfrac{1}{8}x^2$

$x^2 + 24x - 256 = 0$

$(x + 32)(x - 8) = 0$

As $x > 0$

$x = 8$

13 $a = \frac{15}{4x^2} = \frac{15}{4}x^{-2}$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{15}{4}x^{-2}$$

$$\frac{1}{2}v^2 = \int \frac{15}{4}x^{-2}\,dx = -\frac{15}{4}x^{-1} + A$$

$$\frac{1}{2}v^2 = A - \frac{15}{4x}$$

At $x = 5,\ v = -6$

$$18 = A - \frac{15}{20} \Rightarrow A = 18\frac{3}{4} = \frac{75}{4}$$

$$\frac{1}{2}v^2 = \frac{75}{4} - \frac{15}{4x} = \frac{15}{4}\left(5 - \frac{1}{x}\right)$$

When $v = 0$

$$5 - \frac{1}{x} = 0 \Rightarrow x = \frac{1}{5}$$

14 a $a = 3 - x$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 3 - x$$

$$\frac{1}{2}v^2 = \int (3 - x)\,dx = 3x - \frac{x^2}{2} + A$$

$$v^2 = B + 6x - x^2, \text{ where } B = 2A$$

At $x = 0,\ v = 4$

$$16 = B + 0 - 0 \Rightarrow B = 16$$

$$v^2 = 16 + 6x - x^2$$

b $v^2 = 16 + 6x - x^2 = 25 - 9 + 6x - x^2$

$$= 25 - (x-3)^2$$

As $(x-3)^2 \geqslant 0,\ v^2 \leqslant 25$

The greatest value of v is 5.

15 a $a = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = \frac{x^2(5-x)}{2} = \frac{5x^2}{2} - \frac{x^3}{2}$

$\frac{1}{2}v^2 = \int\left(\frac{5x^2}{2} - \frac{x^3}{2}\right)\mathrm{d}x = \frac{5x^3}{6} - \frac{x^4}{8} + A$

$v^2 = \frac{5x^3}{3} - \frac{x^4}{4} + B$, where $B = 2A$

At $x = 10$, $v = 0$

$0 = \frac{5000}{3} - \frac{10\,000}{4} + B \Rightarrow B = \frac{2500}{3}$

$v^2 = \frac{5x^3}{3} - \frac{x^4}{4} + \frac{2500}{3}$

b When $t = 0$, $x = 0$

$v^2 = \frac{2500}{3} \Rightarrow v = (\pm)\frac{50}{\sqrt{3}} = (\pm)\frac{50\sqrt{3}}{3}$

The speed of P when $t = 0$ is $\frac{50\sqrt{3}}{3}\,\mathrm{m\,s^{-1}}$.

16 a $a = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = \frac{20}{5x+2}$

$\frac{1}{2}v^2 = \int\frac{20}{5x+2}\mathrm{d}x = 4\ln(5x+2) + A$

$v^2 = 8\ln(5x+2) + B$, where $B = 2A$

At $x = 0$, $v = 3$

$9 = 8\ln 2 + B \Rightarrow B = 9 - 8\ln 2$

$v^2 = 8\ln(5x+2) - 8\ln 2 + 9 = 8\ln\left(\frac{5x+2}{2}\right) + 9$

At $x = 12$

$v^2 = 8\ln 31 + 9 = 36.471\ldots$

$v = \sqrt{36.471\ldots} = 6.039...$

The speed of P at $x = 12$ is 6.04 m s^{-1} (3 s.f.).

b When $v = 5$

$25 = 8\ln\left(\frac{5x+2}{2}\right) + 9$

$\ln\left(\frac{5x+2}{2}\right) = \frac{25-9}{8} = 2$

$\frac{5x+2}{2} = \mathrm{e}^2$

$x = \frac{2\mathrm{e}^2 - 2}{5} = 2.56\ (3\,\text{s.f.})$

17 a $a = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = 4 + \dfrac{1}{2}x$

$\dfrac{1}{2}v^2 = 4x + \dfrac{x^2}{4} + A$

$v^2 = 8x + \dfrac{x^2}{2} + B$, where $B = 2A$

At $x = 0$, $v = 3$

$9 = 0 + 0 + B \Rightarrow B = 9$

$v^2 = 8x + \dfrac{x^2}{2} + 9$

At $x = 4$

$v^2 = 32 + 8 + 9 = 49 \Rightarrow v = 7$

The speed of P at $x = 4$ is 7 m s^{-1}.

b $a = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = -4 - \dfrac{1}{2}x$

$\dfrac{1}{2}v^2 = C - 4x - \dfrac{x^2}{4}$

$v^2 = D - 8x - \dfrac{x^2}{2}$, where $D = 2C$

At $x = 4$, $v = 7$

$49 = D - 32 - 8 \Rightarrow D = 89$

$v^2 = 89 - 8x - \dfrac{x^2}{2}$

When $v = 0$

$x^2 + 16x = 178 \Rightarrow x^2 + 16x + 64 = 242$

$(x+8)^2 = 242 \Rightarrow x = 11\sqrt{2} - 8$, as $x > 0$

$x = 7.56$ (3 s.f.)

18 a $a = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = 4x + 6$

$\dfrac{1}{2}v^2 = 2x^2 + 6x + A$

$v^2 = 4x^2 + 12x + B$, where $B = 2A$

At $x = 0$, $v = 3$

$9 = 0 + 0 + B \Rightarrow B = 9$

$v^2 = 4x^2 + 12x + 9 = (2x+3)^2$

As v is increasing as x increases

$v = 2x + 3$

18 b $v = \frac{dx}{dt} = 2x + 3$

Separating the variable and integrating

$$\int \frac{1}{2x+3} dx = \int 1 dt$$

$$\frac{1}{2}\ln(2x+3) = t + C$$

$$\ln(2x+3) = 2t + 2C$$

$$2x + 3 = e^{2t+2C} = De^{2t}, \text{ where } D = e^{2C}$$

When $t = 0$, $x = 0$

$$3 = De^{0} \Rightarrow D = 3$$

$$2x + 3 = 3e^{2t}$$

$$x = \frac{3}{2}(e^{2t} - 1)$$

Challenge

Initial velocity $= 32\,500 \text{ km h}^{-1} = \frac{32\,500}{60^2} \text{ km s}^{-1} = 9.0278 \text{ km s}^{-1}$

$$a = v\frac{dv}{dx} = -\frac{c}{x^2}$$

Separating the variables and integrating

$$\int v\,dv = -\int cx^{-2}\,dx$$

$$\frac{1}{2}v^2 = cx^{-1} + A$$

When $x = 6370$, $v = 9.028$

$$\frac{1}{2} \times 81.5008 = \frac{4}{6370} \times 10^5 + A$$

$$A = 40.7504 - 62.7943 = -22.0439$$

When $v = 0$

$$cx^{-1} = -A$$

$$x = -\frac{c}{A} = \frac{4}{22.0439} \times 10^5 = 1.81456 \times 10^4 = 18\,145.6$$

So distance above the Earth's surface $= 18\,145.6 - 6370 = 11\,776$ km (5 s.f.)